DIFFERENTIAL VECTOR CALCULUS

Math
Physics

DIFFERENTIAL VECTOR CALCULUS
by
R. D. Young

1. Introduction ................................................. 1
2. Procedures ................................................... 1
3. Supplementary Notes ........................................ 2
Acknowledgments ................................................. 4
Title: Differential Vector Calculus
Author: R. D. Young, Dept. of Physics, Illinois State Univ.
Version: 10/18/2001 Evaluation: Stage B0
Length: 2 hr; 8 pages

Input Skills:
1. Vocabulary: electrostatic potential field \( \phi(\vec{r}) \), Maxwell’s equations, magnetic induction, electric field, free charge density, and current density.

Output Skills (Knowledge):
K1. Write down the definition or explanation of each of the following terms and concepts: del operator \( \vec{\nabla} \), gradient (of a scalar field), geometric interpretation of gradient, divergence of a vector field, curl of a vector field, geometric meaning of divergence, Laplacian of scalar field, Laplace’s equation \( \nabla^2 \phi = 0 \).
K2. Verify simple vector identities involving the del operator, including successive applications of del.

Output Skills (Rule Application):
R1. Calculate the gradient and Laplacian of a scalar field given the analytic form of the scalar field.
R2. Calculate the divergence and curl of a vector field given the analytic form of the vector field.

External Resources (Required):

This is a developmental-stage publication of Project PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

PROJECT STAFF
Andrew Schneppe Webmaster
Eugene Kales Graphics
Peter Signell Project Director

ADVISORY COMMITTEE
D. Alan Bromley Yale University
E. Leonard Jossem The Ohio State University
A. A. Strassenburg S. U. N. Y., Stony Brook

Views expressed in a module are those of the module author(s) and are not necessarily those of other project participants.

© 2001, Peter Signell for Project PHYSNET, Physics-Astronomy Bldg., Mich. State Univ., E. Lansing, MI 48824; (517) 355-3784. For our liberal use policies see:

DIFFERENTIAL VECTOR CALCULUS

by

R. D. Young

1. Introduction

The del operator \( \vec{\nabla} \) which is introduced in this unit is a differential vector operator. Del must satisfy all the rules for vectors and partial differentiation. The concepts of the gradient of a scalar field and the divergence and curl of a vector field are central to all of theoretical physics from Newtonian mechanics to quantum field theory. The gradient of a scalar field has occurred already in your career in physics as an undergraduate. The electrostatic force \( \vec{F} (\vec{r}) \) on a particle of charge \( q \) at position \( \vec{r} \) in an electrostatic potential \( \phi (\vec{r}) \) is just given by:

\[
\vec{F} = -q \vec{\nabla} \phi,
\]

where \( \vec{\nabla} \phi \) is “the gradient of \( \phi \)” or simply “del \( \phi \).” The operations of divergence and curl have been seen already also. For example, in vacuum with no charges or currents present in a region, the electric field \( \vec{E} (\vec{r}, t) \) and magnetic induction \( \vec{B} (\vec{r}, t) \) satisfy these equations:

\[
\begin{align*}
\vec{\nabla} \cdot \vec{E} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\partial \vec{B}/\partial t \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{B} &= \mu_0 \epsilon_0 \partial \vec{E}/\partial t.
\end{align*}
\]

Here \( \vec{\nabla} \cdot \vec{E} \) is “the divergence of \( E \)” and \( \vec{\nabla} \times \vec{E} \) is “the curl of \( E \).” In the introduction to MISN-0-478, we saw that temperature \( T \) can be a scalar field \( T (\vec{r}, t) \) which also can depend on time. In a later unit, we shall see that the flow of heat is determined by \( \vec{\nabla} T \). These are just a few instances where del, \( \vec{\nabla} \), plays an important role in the mathematical analysis of physical phenomena.

2. Procedures

1. Read pages 125 - 127 of Spiegel. Begin with the section on “Limits, Continuity, etc.” Do not include the section on “Orthogonal Curvilinear Coordinates” on page 127.
The electrostatic field can always be derived from a scalar potential $\phi(\vec{r})$ as

$$\vec{E} = -\nabla \phi(\vec{r}).$$

It should be noted that problem 1.11.6 and 1.12.4 in Arfken gives boundary conditions on the displacement $D$ and magnetic intensity $H$, respectively, due to the presence of free charge and current at a boundary.

**Application of Del in Products**

The discussion on page 46 of Arfken including example 1.8.2 is meant to illustrate a method which will shorten the calculation of vector identities in which del appears in a product. The idea is to use the regular BAC-CAB rule keeping in mind that del also must satisfy partial differentiation rules. As an example, problem 1.8.12 will be solved.

1.8.12 Show that

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \times \nabla) \times \vec{B} + (\vec{B} \times \nabla) \times \vec{A} + \vec{A}(\nabla \cdot \vec{B}) + \vec{B}(\nabla \cdot \vec{A}).$$

Solution: Consider the quantity $(\vec{A} \times \nabla_B) \times \vec{B}$ where the subscript $B$ has been attached to $\nabla$ to show that it only differentiates $\vec{B}$. Applying the BAC-CAB rule by rewriting the triple cross product but rearranging the terms so $\nabla_B$ always appears to the left of $\vec{B}$ in the final result gives

$$(\vec{A} \times \nabla_B) \times \vec{B} = -\vec{B} \times (\vec{A} \times \nabla_B) = -[\vec{A}(\nabla_B \cdot \vec{B}) - \nabla_B(\vec{A} \cdot \vec{B})].$$

In the same way,

$$(\vec{B} \times \nabla_A) \times \vec{A} = -\vec{A} \times (\vec{B} \times \nabla_A) = -[\vec{B}(\nabla_A \cdot \vec{A}) - \nabla_A(\vec{A} \cdot \vec{B})].$$

Adding gives

$$(\vec{A} \times \nabla) \times \vec{B} + (\vec{B} \times \nabla) \times \vec{A} = \nabla_A(\vec{A} \cdot \vec{B}) + \nabla_B(\vec{A} \cdot \vec{B}) - \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A})$$

where the subscripts $A$ and $B$ have been dropped wherever no ambiguity exists concerning which vector is being differentiated by del. Thus

$$\nabla_A(\vec{A} \cdot \vec{B}) + \nabla_B(\vec{A} \cdot \vec{B}) = (\vec{A} \times \nabla) \times \vec{B} + (\vec{B} \times \nabla) \times \vec{A} + \vec{A}(\nabla \cdot \vec{B}) + \vec{B}(\nabla \cdot \vec{A}).$$

**Acknowledgments**

The author would like to thank Illinois State University for support in the construction of this lesson. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.