COMPUTER ALGORITHM FOR THE DAMPED DRIVEN OSCILLATOR

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Input Skills:
1. Expand a given function about a given point using Taylor’s series (MISN-0-4).

Output Skills (Knowledge):
K1. Given the force acting on a damped driven oscillator along with the oscillator’s position and velocity at a specified time, derive a Numerov type algorithm for the approximate numerical calculation of the oscillator’s position at all past and future times.

Post-Options:
1. “Response of a Damped Driven Oscillator” (MISN-0-30).
2. “Damped Driven Oscillations; Mechanical Resonances” (MISN-0-31).

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COMPUTER ALGORITHM FOR THE DAMPED DRIVEN OSCILLATOR
by
Peter Signell

1. Derivation

1a. Equation for a Damped Driven Oscillator. The equation to be solved is that of the damped harmonically-driven oscillator acted on by the force:

\[ F(t) = -kx(t) - \lambda v(t) + F_0 \cos(\omega t). \]

Since

\[ F(t) = ma(t) = m \frac{dx(t)}{dt} \equiv mx''(t), \]

and

\[ v(t) = \frac{dx(t)}{dt} \equiv x'(t), \]

our equation to be solved is:

\[ mx''(t) + kx(t) + \lambda x'(t) - f(t) = 0, \]

where

\[ f(t) \equiv F_0 \cos(\omega t). \]

1b. Approximate Derivatives by Finite Differences. We now make two power series expansions about time t:

\[ x(t + \Delta) = x(t) + \Delta x'(t) + (\Delta^2/2)x''(t) + (\Delta^3/6)x'''(t) + \ldots \]

\[ x(t - \Delta) = x(t) - \Delta x'(t) + (\Delta^2/2)x''(t) - (\Delta^3/6)x'''(t) + \ldots \]

We will choose \( \Delta \) sufficiently small so that we can disregard all terms after \( \Delta^2 \) without incurring much error. Then add and subtract the above equations to obtain:

\[ x(t + \Delta) + x(t - \Delta) \approx 2x(t) + \Delta^2 x''(t) \]

1c. Net-Point Times. We need a more succinct labeling system at this point so we won’t be able to see the forest for the trees. We define discrete “net-point” times as: \( t_n \equiv n\Delta \), where \( \Delta \) is some small time interval (see Fig.1). We can then rewrite the derivatives at time \( t_n \) as:

\[ x''(t) \approx \frac{1}{\Delta^2} (x_{n+1} - 2x_n + x_{n-1}) \]

\[ x'(t) \approx \frac{1}{2\Delta} (x_{n+1} - x_{n-1}) \]

which are often quoted in calculus courses.

Then at time \( t_n \) our basic Eq. (1) becomes:

\[ mx''(t) + kx_n + \lambda x'_n - f_n = 0. \]

Substituting for \( x''_n \) and \( x'_n \):

\[ m \frac{\Delta^2}{\Delta^2} (x_{n+1} - 2x_n + x_{n-1}) + kx_n + \frac{\lambda}{2\Delta} (x_{n+1} - x_{n-1}) - f_n = 0 \]

The second equation above is used in “Response of a Damped Driven Oscillator” (MISN-0-30) where a microcomputer is utilized in finding solutions.
Collecting coefficients of the $x$ values, we get:

$$x_{n+1} = Ax_n + Bx_{n-1} + \Delta^2 C_n$$

where:

$$A = \left(2 - \Delta^2 \frac{k}{m}\right)/D$$

$$B = -\left(1 - \Delta \frac{\lambda}{2m}\right)/D$$

$$C_n = \left(\frac{f_n}{m}\right)/D$$

$$D = 1 + \Delta \frac{\lambda}{2m}$$

### 2. Application

#### 2a. Iterating the Recurrence Relation.

Equation (4) is called a three point recurrence relation because it relates $x$ at three successive time net-points. If you know $x_0$ and $x_1$, you can use this relation to calculate $x_2$. Then knowing $x_1$ and $x_2$, you can calculate $x_3$, etc. You can keep up this recurrence procedure until you have reached the time for which you wish to know the solution. For example, this might be a time sufficiently large so that transient effects have died away and only the steady state solution remains.

\[ \text{Figure 2. The Calculated Curve. Draw a smooth curve through the plotted points and you have } x(t). \]

#### 2b. Starting the Iteration.

With a three-point recurrence relation one can calculate values at successive time net-points, but one needs two adjacent values in order to get started. In the example cited in the preceding paragraph, one needed $x_0$ and $x_1$ in order to start the procedure. Those two initial values generally come from a specification of a “complete set of initial conditions.” Our Eq. (1), being a second order equation for $x$, always requires that one specify two independent conditions on $x$.

A common case is where both the position and velocity are known at some instant of time. Here we will call that time $t = 0$; hence we know $x_0$ and $x_1$ by solving Eq. (3) at $n = 0$ for $x_{-1}$ (Note: $v_0 = x'_0$), and using that to eliminate $x_{-1}$ from Eq. (4) for $n = 0$. We thus obtain $x_1$ in terms of $x_0$ and $v_0$ (usually given quantities):

$$x_1 = \left(1 - \Delta^2 \frac{k}{2m}\right)x_0 + \left(\Delta - \Delta^2 \frac{\lambda}{2m}\right)v_0 + \Delta^2 \frac{f_0}{2m}$$

Now the algorithm is complete. Put in $x_0$ and $v_0$ to get $x_0$ and $x_1$, $x_0$ and $x_1$ to get $x_2$, $x_1$ and $x_2$ to get $x_3$, etc.

┚ How would you obtain position values for times earlier than the one for which $x$ and $v$ are initially known? Answer: To see the answer, replace each of the following letters by its successor in the alphabet: mdfzshud cdksz.

### Acknowledgments

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MODEL EXAM

1. The force acting on a damped driven oscillator is \( F = -kx - \lambda v + F_0 \cos(\omega t) \); the oscillator’s initial position and speed are \( x_0 \) and \( v_0 \). Derive without notes, a complete Numerov-type algorithm for the (approximate) calculation of the oscillator’s position at all times. NOTE: The algorithm is to include the method of use of any three point recurrence relation such as:

\[
x_{n+1} = Ax_n + Bx_{n-1} + \Delta^2 C_n
\]

where:

\[
A = \left( 2 - \Delta^2 \frac{k}{m} \right) / D
\]

\[
B = -\left( 1 - \Delta \frac{\lambda}{2m} \right) / D
\]

\[
C_n = \left( \frac{f_n}{m} \right) / D
\]

\[
D = 1 + \Delta \frac{\lambda}{2m}
\]

and boundary conditions such as:

\[
x_1 = \left( 1 - \Delta^2 \frac{k}{2m} \right) x_0 + \left( \Delta - \Delta^2 \frac{\lambda}{2m} \right) v_0 + \Delta^2 \frac{f_0}{2m}
\]

Brief Answers:

1. See this module’s text.